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Parameters of Association Schemes That Are Both P- and Q-Polynomial

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1. INTRODUCTION

Bannai in his recent lectures [1] has proposed the task of characterizing all association schemes that are both P- and Q-polynomial in the sense of Delsarte [2], believing that he has an essentially complete list at this time (except for those with small diameter, a class containing all strongly regular graphs among others). Egawa [3] has the first characterization result along this path.

In general a d-class scheme that is only P-polynomial (metric) could have as many as $2d-1$ independent parameters. This is far from true in the case of schemes that are both P- and Q-polynomial.

In this paper, we show

MAIN THEOREM. *If (X, R) is an association scheme that is both P- and Q-polynomial, then regardless of the number d of classes, it suffices to know only the parameters*

$$K = p_{10}^1, \qquad a_1 = p_{11}^1, \qquad a_2 = p_{22}^1, \qquad c_2 = p_{12}^1, \qquad \theta_1, K^*, a_0^*, c_1^*$$

where θ_1 is the first eigenvalue under the ordering imposed by the Q-polynomials and K^* is its multiplicity.

We further conjecture that, since θ_1 is at most quadratic over $\mathbb{Z}$, it can be replaced by two further integer parameters, namely, either K_2^* and K_3^* or a_3 and c_3.

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2. Preliminaries

We shall call $\underline{p} = (p_j)_{j=0}^d$ a *graded sequence of polynomials* iff $\deg p_j = j$ for all j. For our purposes $\underline{p}$ is a *sequence of orthogonal polynomials* iff it is a graded sequence satisfying some three-term recurrence relation.

Lemma 1. *Let $\underline{p}$ be a graded sequence of polynomials. Suppose $f_i = \sum_{j=0}^d h_{ji} p_j$ for $i = 0,\ldots, t$ are linear combinations of these polynomials having common zeros $x_1,\ldots, x_s$. Let*

$$h_j = \begin{vmatrix} h_{d,0} & \cdots & h_{d,t} \\ \vdots & & \vdots \\ h_{d-t+1,0} & \cdots & h_{d-t+1,t} \\ h_{j,0} & \cdots & h_{j,t} \end{vmatrix}.$$

Then $f = \sum_{j=0}^d h_j p_j$ is a polynomial of degree at most $d - t$ having the zeros $x_1,\ldots, x_s$.

Proof. The polynomial f is a linear combination of $f_0,\ldots, f_t$, so it is necessarily zero at $x_1,\ldots, x_s$. For $j > d - t$, the last row of h_j is identical to some previous row, so $h_j = 0$ for $j > d - t$. Hence f has degree at most $d - t$.

Lemma 2. *Let $\underline{p}$ be a graded sequence of polynomials and $\underline{x} = (x_i)_{i=0}^d$ a sequence of distinct real numbers. Then*

$$M_d(\underline{p}, x) = (p_j(x_i))$$

is a non-singular $(d + 1) \times (d + 1)$ matrix.

Proof. Apply the obvious column operations to reduce $M_n(\underline{p}, \underline{x})$ to a van der Monde matrix.

3. P- and Q-Polynomiality

We shall try to describe as briefly as possible the relevant theory of association schemes, but suggest Delsarte's original work [2] or the relevant section of MacWilliams–Sloane [4] to complete this description.

Given an association scheme (X, R) there are two important matrices $P = (p_j(i))$ and $Q = (q_j(i)) = |X| \cdot P^{-1}$ also related by $p_j(i)\, q_i(0) = p_j(0)\, q_i(j)$ where $p_j(0) = |R_j|$ and $q_i(0)$ is the multiplicity of the ith eigenvalue of R_1. These matrices are related to the change of bases from the adjacency matrices of the scheme to the primitive idempotents of the algebra they generate.

The entries of the P matrix are interrelated by the connection parameters

$$p^i_{jk} = |\{z: xR_i z \text{ and } zR_j y\}|$$

which are independent of the choice of x, y such that $xR_k y$. These are determined by

$$p_i(s)\, p_j(s) = \sum_{k=0}^{d} p^i_{jk}\, p_k(s).$$

Similarly there are dual (Krein) parameters q^i_{jk} determined by

$$q_i(s)\, q_j(s) = \sum_{k=0}^{d} q^i_{jk} q_k(s).$$

An association scheme is called *P-polynomial* iff there exists a sequence (v_j) of orthogonal polynomials and a sequence (θ_i) of distinct real numbers such that $p_j(i) = v_j(\theta_i)$ for all i, j. Delsarte [2] showed this to be equivalent to metric regularity of the adjacency graph for one of the relations R_i. Hence, the relations may be ordered in such a way that the p^i_{jk}'s satisfy the triangle inequality; that is, $p^i_{jk} = 0$ if $k \notin [|j-i|, i+j]$.

Further all the p^i_{jk}'s can be determined by the p^1_{ij}'s, so the matrix

$$B = (p^1_{ij}) = \begin{pmatrix} 0 & 1 & & & & \\ K & a_1 & c_2 & & & \\ & b_1 & a_2 & & & \\ & & b_2 & & & \\ & & & \ddots & & \\ & & & & & c_d \\ & & & & b_{d-1} & a_d \end{pmatrix}$$

plays an important role. It gives the orthogonality relations for the polynomials (v_j); namely,

$$v_0(\lambda) = 1, \qquad v_1(\lambda) = \lambda, \qquad c_{n+1} v_{n+1}(\lambda) = (\lambda - a_n)\, v_n(\lambda) - b_{n-1} v_{n-1}(\lambda),$$

and, for eigenvalues θ of B, the additional relation

$$b_{d-1} v_{d-1}(\theta) = (\theta - a_d)\, v_d(\theta).$$

It is possible to use this last equation and the recurrence relations to write $v_j(\theta)/v_d(\theta)$ as a polynomial of degree $d-j$ in θ, $u_{d-j}(\theta)$ say.

Since the columns of B all sum to $K = |R_1|$, there are at most $2d-1$ independent parameters for P-polynomial schemes. These must satisfy the additional restrictions that $\mu_j = q_j(0)$ and $K_j = p_j(0)$ be integral , that the c_i's and b_i's be monotone, and that the Krein parameters q^i_{jk} be non-negative (the Krein condition). While these are all non-trivial constraints, they do not limit the size of the P-polynomial scheme characterization problem appreciably.

An association scheme is called *Q-polynomial* iff there exists a sequence (z_j) of orthogonal polynomials and a sequence (v_i) of distinct real numbers such that $q_j(i) = z_j(v_i)$. There is as yet no equivalent theorem to Delsarte's, describing the importance of Q-polynomiality.

Note, however, that if a dual scheme exists, this is merely a requirement that it be P-polynomial. (In light of this, the terms metric and dual metric might be more descriptive than P-polynomial and Q-polynomial.) P-polynomiality induces an ordering on the columns of the P matrix (and hence the rows of the Q matrix). Dually Q-polynomiality induces an ordering on the columns of the Q matrix (and hence the rows of the P matrix).

The proof we give below on determining parameters is completely dual in nature and does not rely on the existence of an underlying association scheme. We assume only that we have a tridiagonal matrix B as above which defines a sequence of orthogonal polynomials (v_j). We shall then say that B is *polynomial* with associated matrix $P(B, \pi) = (v_j(\theta_i))$, where $\theta_0 = K$ and π is some ordering on the remaining eigenvalues of B. Let $P^*(B, \pi) = (1/n)\, P^{-1}(B, \pi)$, where $n = \sum_{j=0}^{d} v_j(K)$. If for some ordering π there exists a polynomial matrix B^* defining the sequence (v_j^*) of orthogonal polynomials, such that $P^*(B, \pi) = P(B^*, \pi^*)$ for some ordering π^* of the eigenvalues of B^*, then B will be called *dual polynomial.*

Thus in Delsarte's terminology, a P-polynomial association scheme is Q-polynomial iff the associated B matrix is dual polynomial.

From now on we assume B is polynomial and dual polynomial and that $d \geqslant 3$ (meaning b_1, b_2, b_1^*, b_2^* are all positive), since all association schemes with $d \leqslant 2$ are necessarily both P- and Q-polynomial. We shall also use dual notation throughout. For instance q^i_{jk} will now be written $p^i_{jk}{}^*$.

It is always possible to solve for the Krein parameters in terms of the entries in the P-matrix. Under our assumption of polynomiality and dual polynomiality this becomes

$$p^i_{jk}{}^* = \frac{v_j^*(K^*)\, v_i^*(K^*)}{n} \cdot \sum_{l=0}^{d} \frac{v_l(\theta_i)\, v_l(\theta_j)\, v_l(\theta_k)}{v_l^2(K)}.$$

Since $v_j^*(K^*)\, v_i^*(K^*)/n$ is positive, we shall concentrate on the summation term. We try to exploit the fact that because of dual polynomiality $p^i_{jk}{}^*$ must be zero whenever i, j, k do not satisfy the triangle inequality. Let

$$A_{j,k} = \frac{v_j(\theta_k)}{v_j(K)},$$

$$a_{ijk}=\sum_{l=0}^{d}\frac{v_l(\theta_i)\,v_l(\theta_j)\,v_l(\theta_k)}{v_l^2(K)}=\sum_{l=0}^{d}A_{li}A_{lj}v_l(\theta_k),$$

$$a_{ij}(x)=\sum_{l=0}^{d}\frac{v_l(\theta_i)\,v_l(\theta_j)}{v_l^2(K)}v_l(x)=\sum_{l=0}^{d}A_{li}A_{lj}v_l(x).$$

Then a_{ij} is a polynomial of degree at most d, and is zero for all θ_k with $k\notin[|j-i|,i+j]$ by the triangle inequality.

4. Proof of Main Theorem

The sequence (v_j) is a graded sequence of polynomials and the a_{ij}'s are linear combinations of them. So for $m>1$, apply Lemma 1 to the polynomials a_{00}, a_{0m-1}, a_{0m}, a_{0m+1}, and $a_{1,m}$. The resulting polynomial f has degree at most $(d+1)-5$ and zeros $\theta_1,\ldots,\theta_{m-2},\theta_{m+2},\ldots,\theta_d$. Hence f must be identically zero, which means

$$\begin{vmatrix}A_{d,0} & A_{d,m-1} & A_{d,m} & A_{d,m+1} & A_{d,1} & A_{d,m}\\ A_{d-1,0} & A_{d-1,m-1} & A_{d-1,m} & A_{d-1,m+1} & A_{d-1,1} & A_{d-1,m}\\ A_{d-2,0} & A_{d-2,m-1} & A_{d-2,m} & A_{d-2,m+1} & A_{d-2,1} & A_{d-2,m}\\ A_{d-3,0} & A_{d-3,m-1} & A_{d-3,m} & A_{d-3,m+1} & A_{d-3,1} & A_{d-3,m}\\ A_{j,0} & A_{j,m-1} & A_{j,m} & A_{j,m+1} & A_{j,1} & A_{j,m}\end{vmatrix}=0$$

for all j. View this as linear dependence relations among $(d+1)$ row vectors. By applying Lemma 2 to the upper left 4×4 matrix, we can see that the first four rows are linearly independent. So any five rows are linearly dependent. In particular

$$\begin{vmatrix}A_{0,0} & A_{0,m-1} & A_{0,m} & A_{0,m+1} & A_{0,1} & A_{0,m}\\ A_{1,0} & A_{1,m-1} & A_{1,m} & A_{1,m+1} & A_{1,1} & A_{1,m}\\ A_{2,0} & A_{2,m-1} & A_{2,m} & A_{2,m+1} & A_{2,1} & A_{2,m}\\ A_{3,0} & A_{3,m-1} & A_{3,m} & A_{3,m+1} & A_{3,1} & A_{3,m}\\ A_{j,0} & A_{j,m-1} & A_{j,m} & A_{j,m+1} & A_{j,1} & A_{j,m}\end{vmatrix}=0.$$

Multiply the last row by $v_j(K)$ and sum over j to get

$$\begin{vmatrix}A_{0,0} & A_{0,m-1} & A_{0,m} & A_{0,m+1} & A_{0,1}A_{0,m}\\ A_{1,0} & A_{1,m-1} & A_{1,m} & A_{1,m+1} & A_{1,1}A_{1,m}\\ A_{2,0} & A_{2,m-1} & A_{2,m} & A_{2,m+1} & A_{2,1}A_{2,m}\\ A_{3,0} & A_{3,m-1} & A_{3,m} & A_{3,m+1} & A_{3,1}A_{3,m}\\ a_{000} & a_{00m-1} & a_{00m} & a_{00m+1} & a_{01m}\end{vmatrix}=0.$$

But $a_{00m-1}=a_{00m}=a_{00m+1}=a_{01m}=0$ by the triangle inequality and $a_{000}=p_{00}^{0*}(n/(v_0^*(K^*))^2=n$. So we have

$$\begin{vmatrix} A_{0,0} & A_{0,m-1} & A_{0,m} & A_{0,m+1} & A_{0,1}A_{0,m} \\ A_{1,0} & A_{1,m-1} & A_{1,m} & A_{1,m+1} & A_{1,1}A_{1,m} \\ A_{2,0} & A_{2,m-1} & A_{2,m} & A_{2,m+1} & A_{2,1}A_{2,m} \\ A_{3,0} & A_{3,m-1} & A_{3,m} & A_{3,m+1} & A_{3,1}A_{3,m} \\ n & 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

View this as an equation of degree at most three in the variable θ_{m+1}. By comparing the fourth column with the second and third columns, we can see two obvious solutions $\theta_{m+1}=\theta_m$ and $\theta_{m+1}=\theta_{m-1}$. But since B has distinct eigenvalues we get

$$\begin{aligned} &b_2(\theta_{m+1}+\theta_m+\theta_{m-1}-a_1-a_2) \\ &\qquad\times[(\theta_1+K-a_1)\,V_2(\theta_m)-(\theta_{m-1}+\theta_m-a_1)\,\theta_m b_1] \\ &\quad=\frac{V_3(\theta_1)-V_3(K)}{\theta_1-K}\,V_3(\theta_m)-\frac{V_3(\theta_{m-1})-V_3(\theta_m)}{\theta_{m-1}-\theta_m}\,\theta_m b_1 b_2 \end{aligned}$$

where $V_j(x)=(\prod_{l=1}^{j} c_l)\,v_j(x)$ is monic. This expression allows us to solve recursively for θ_j in terms of θ_1 and θ_2 provided that the term in brackets on the left-hand side is not zero.

Had we multiplied by $v_j(\theta_{m-1})$, $v_j(\theta_m)$, or $v_j(\theta_{m+1})$ instead of $v_j(K)$ and summed, we would have gotten last rows of

$$a_{00,m-1} \quad a_{0,m-1,m-1} \quad a_{0,m-1,m} \quad a_{0,m-1,m+1} \quad a_{1,m-1,m},$$

$$a_{00m} \quad a_{0,m-1,m} \quad a_{0mm} \quad a_{0m,n+1} \quad a_{1mm},$$

or

$$a_{00,m+1} \quad a_{0,m-1,m+1} \quad a_{0m,m+1} \quad a_{0,m+1,m+1} \quad a_{1m,m+1}$$

respectively.

All of these are zero, by the triangle inequality, except

$$a_{0,m-1,m-1}=p_{m-1,m-1}^{0\;*}\frac{n}{v_0^*(K^*)\,v_{m-1}^*(K^*)}=\frac{n}{v_{m-1}^*(K^*)},$$

$$a_{1,m-1,m}=p_{m-1,m}^{1\;*}\frac{n}{v_1^*(K^*)\,v_{m-1}^*(K^*)},$$

$$a_{0mm}=p_{mm}^{0\;*}\frac{n}{v_0^*(K^*)\,v_m^*(K^*)}=\frac{n}{v_m^*(K^*)},$$

$$a_{1mm} = p^{1}_{mm}{}^{*} \frac{n}{v_1^*(K^*)\, v_m^*(K^*)},$$

$$a_{0,m+1,m+1} = p^{0}_{m+1,m+1}{}^{*} \frac{n}{v_0^*(K^*)\, v_{m+1}^*(K^*)} = \frac{n}{v_{m+1}^*(K^*)},$$

$$a_{1m,m+1} = p^{1}_{m+1,m}{}^{*} \frac{n}{v_1^*(K^*)\, v_{m+1}^*(K^*)}.$$

So, properly scaled, these last rows become

$$\begin{array}{ccccc} 0 & v_1^*(K^*) & 0 & 0 & p^{1}_{m-1,m}{}^{*}, \\ 0 & 0 & v_1^*(K) & 0 & p^{1}_{mm}{}^{*}, \\ 0 & 0 & 0 & v_1^*(K^*) & p^{1}_{m+1,m}{}^{*}, \end{array}$$

from which

$$c_m^* = p^{1}_{m-1,m}{}^{*} = \frac{v_1^*(K^*)}{A_m} \cdot \begin{vmatrix} 1 & A_{0,m} & A_{0,m+1} & A_{0,1}A_{0,m} \\ 1 & A_{1,m} & A_{1,m+1} & A_{1,1}A_{1,m} \\ 1 & A_{2,m} & A_{2,m+1} & A_{2,1}A_{2,m} \\ 1 & A_{3,m} & A_{3,m+1} & A_{3,1}A_{3,m} \end{vmatrix},$$

$$a_m^* = p^{1}_{mm}{}^{*} = \frac{v_1^*(K^*)}{A_m} \cdot \begin{vmatrix} 1 & A_{0,m+1} & A_{0,m-1} & A_{0,1}A_{0,m} \\ 1 & A_{1,m+1} & A_{1,m-1} & A_{1,1}A_{1,m} \\ 1 & A_{2,m+1} & A_{2,m-1} & A_{2,1}A_{2,m} \\ 1 & A_{3,m+1} & A_{3,m-1} & A_{3,1}A_{3,m} \end{vmatrix},$$

$$b_m^* = p^{1}_{m+1,m}{}^{*} = \frac{v_1^*(K^*)}{A_m} \begin{vmatrix} 1 & A_{0,m-1} & A_{0,m} & A_{0,1}A_{0,m} \\ 1 & A_{1,m-1} & A_{1,m} & A_{1,1}A_{1,m} \\ 1 & A_{2,m-1} & A_{2,m} & A_{2,1}A_{2,m} \\ 1 & A_{3,m-1} & A_{3,m} & A_{3,1}A_{3,m} \end{vmatrix},$$

where

$$A_m = \begin{vmatrix} 1 & A_{0,m-1} & A_{0,m} & A_{0,m+1} \\ 1 & A_{1,m-1} & A_{1,m} & A_{1,m+1} \\ 1 & A_{2,m-1} & A_{2,m} & A_{2,m+1} \\ 1 & A_{3,m-1} & A_{3,m} & A_{3,m+1} \end{vmatrix}$$

which is not zero by Lemma 2.

For $m = 1$, apply Lemma 1 to the polynomials a_{00}, a_{01}, a_{02}, and a_{11} which have common zeros $\theta_3, \ldots, \theta_d$. The resulting polynomial f has degree at most $(d+1) - 4$ so it again must be identically zero, from which

$$\begin{vmatrix} A_{d,0} & A_{d,1} & A_{d,2} & A^2_{d,1} \\ A_{d-1,0} & A_{d-1,1} & A_{d-1,2} & A^2_{d-1,1} \\ A_{d-2,0} & A_{d-2,1} & A_{d-2,2} & A^2_{d-2,1} \\ A_{j,0} & A_{j,1} & A_{j,2} & A^2_{j,1} \end{vmatrix}$$

for all j. As before we can get

$$\begin{vmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A^2_{0,1} \\ A_{1,0} & A_{1,1} & A_{1,2} & A^2_{1,1} \\ A_{2,0} & A_{2,1} & A_{2,2} & A^2_{2,1} \\ A_{j,0} & A_{j,1} & A_{j,2} & A^2_{j,1} \end{vmatrix} = 0$$

for all j. Multiplying by $v_j(K)$, $v_j(\theta_1)$, and $v_j(\theta_2)$, summing and scaling we get last rows of

$$\begin{array}{cccc} v_1^*(K^*) & 0 & 0 & 1, \\ 0 & v_1^*(K^*) & 0 & p^1_{11}{}^*, \\ 0 & 0 & v_1^*(K^*) & p^1_{21}{}^*, \end{array}$$

respectively. Using the first of these, we get an equation of degree at most two in the variable θ_2. It has an obvious solution $\theta_2 = \theta_1$, so

$$b_1(K + v_1^*(K^*)\theta_1)(\theta_2 - K) = v_1^*(K^*)(\theta_1 + K - a_1)(\theta_1 - K)(\theta_1 + 1).$$

From the others we also get

$$c_1^* = p^1_{01}{}^* = \frac{v_1^*(K^*)}{A_1} \begin{vmatrix} A_{0,1} & A_{0,2} & A^2_{0,1} \\ A_{1,1} & A_{1,2} & A^2_{1,1} \\ A_{2,1} & A_{2,2} & A^2_{2,1} \end{vmatrix},$$

$$a_1^* = p^1_{11}{}^* = \frac{v_1^*(K^*)}{A_1} \begin{vmatrix} 1 & A^2_{0,1} & A_{0,2} \\ 1 & A^2_{1,1} & A_{1,2} \\ 1 & A^2_{2,1} & A_{2,2} \end{vmatrix},$$

$$b_1^* = p^1_{21}{}^* = \frac{v_1^*(K^*)}{A_1} \begin{vmatrix} 1 & A_{0,1} & A^2_{0,1} \\ 1 & A_{1,1} & A^2_{1,1} \\ 1 & A_{2,1} & A^2_{1,1} \end{vmatrix},$$

where

$$A_1 = \begin{vmatrix} 1 & A_{0,1} & A_{0,2} \\ 1 & A_{1,1} & A_{1,2} \\ 1 & A_{2,1} & A_{2,2} \end{vmatrix}$$

is not zero by Lemma 2.

To be able to solve for θ_2 above, we need only show that $b_1(K + v_1^*(K^*)\,\theta_1) \neq 0$. So suppose $b_1(K + v_1^*(K^*)\,\theta_1) = 0$. By assumption $b_1 \neq 0$, so $K + v_1^*(K^*)\,\theta_1 = 0$. But

$$\frac{v_1^*(\theta_2^*)}{v_1^*(K^*)} = \frac{v_2(\theta_1)}{v_2(K)} = \frac{\theta_1^2 - a_1\theta - K}{Kb_1}.$$

Also since $K + v_1^*(K)\,\theta_1 = 0$, $(K^*) \neq 0$, θ_1, we must have $\theta_1 + K - a_1 = 0$ or $\theta_1 + 1 = 0$. So substituting for θ_1, we get $\theta_2^* = K^*$ or -1, respectively. But θ_2^* is distinct from K^*. And in the second case $\theta_1^*/K^* = \theta_1/K$, so $\theta_1^* = -1$ and θ_2^* is distinct from θ_1^*.

Now suppose that we could not solve recursively for θ_{m+1}, $m < d$. Then this means that

$$(\theta_1 + K - a_1)\,V_2(\theta_m) - (\theta_{m-1} + \theta_m - a_1)\,\theta_m b_1 = 0$$

and

$$\frac{V_3(\theta_1) - V_3(K)}{\theta_1 - K}\,V_3(\theta_m) - \frac{V_3(\theta_{m-1}) - V_3(\theta_m)}{\theta_{m-1} - \theta_m}\,\theta_m b_1 b_2 = 0$$

since $b_2 \neq 0$ be assumption. From these equations it is possible to show that $b_m^*\theta_m = 0$. But if $b_m^* = 0$ then $m = d$, so we may assume that $\theta_m = 0$. But then $(\theta_1 + K - a_1)(-K) = 0$ so $\theta_1 + K - a_1 = 0$ and $\theta_2^* = K^*$ as above.

We have therefore produce B^* from the parameters a_0^*, c_1^*, K, a_1, a_2, c_2, θ and K^*. Dually we can produce the matrix B since we know K^*, a_1^*, a_2^*, c_2^*, θ_1^*, and K.

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